SUNY ECC

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ACCUPLACER Preparation Workshop

Algebra Skills

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Evaluating Algebraic Expressions

Substitute the value (#) in place of the letter (variable). Follow order of operations!!!

Ex) 2x – 4x + 5; x = 3

$2(3) - 4(3) + 5$

6 - 12 + 5

-6 + 5

-1

Ex) $3n^2(2n-1)+5$; n = -4

$3(-4)^2(2(-4) - 1) + 5$

$3(16)(-8 - 1) + 5$

3(16)(-9) + 5

-432 + 5

-427

Ex) $p^2 - q^2$: p = 5, q = -3

$(5)^2 - (-3)^2$

$25 - 9$

16

Ex) $-w^2 - 5w + 3$; w = 4

$-(4)^2 - 5(4) + 3$

$-16 - 20 + 3$

-36 + 3

-33

Ex) $3(x-4)^2 - (3y-4)^2$; x = -1, y = 2

$3(-1 - 4)^2 - (3(2) - 4)^2$

$3(-5)^2 - (6 - 4)^2$

$3(25) - 2^2$

75 – 4

71

Practice Problems: Evaluate the given expression for the value(s) of the variable(s)

1. **Evaluate** $\boldsymbol{2x^2 + 3x - 4 \; for \; x = -1}$
2. **Evaluate** $\boldsymbol{-x^2 - 4y + 7 \; for \; x = -2 \; and \; y = -3}$
3. **Evaluate** $\boldsymbol{2(x+4)^2 - (3y+5)^3 for \; x = -3 \; and \; y = -1}$

Solutions:

1. -5 2. 15 3. -6

Combining Like Terms

like terms: have exactly the same variable factors

Ex. In the first example above, 2x and 4x are like terms

To simplify an algebraic expression:

1. Remove parenthesis by distributing
2. Combine *like terms* (combine coefficients according to their sign; keep the variables and exponents the same.)

Ex. $4(5y^2-3)+(6y^2+3)$

$20y^2-12+6y^2+3$	Distribute
$26y^2-9$	Combine like terms

Additional Examples. Simplify the following algebraic expressions:

a. 4 – x + 4x – 8 The like terms are –x and 4x; 4 and -8
3x – 4

b. $-2(x^2+3x-7)$
$-2x^2-6x+14$ Distribute. There are no like terms so we are done.

c. $14x^2+5-[7x^2-10]$ Distribute the "-" to eliminate the [ ]
$14x^2+5-7x^2+10$ Combine like terms
$7x^2+15$

d. 8x – 5y – 9x + 4y
-x - y

Practice Problems: Simplify the following expressions:

1. $\boldsymbol{6x-2y+4x-9y-10}$
2. $\boldsymbol{7(9x-8)-3(4y-12)}$
3. $\boldsymbol{-2(x^2+3x-10)+4(x^2+2x)}$

Solutions

1. **10x – 11y – 10**
2. **63x – 12y -20**
3. $\boldsymbol{2x^2+2x+20}$

Solving Linear Equations

The steps to solve a linear equation in one variable can be summarized as follows:

1. Simplify the algebraic expression on *each* side by distributing (if parenthesis are present) and combining like terms.
2. Move all of the terms containing variables to one side and all of the constant terms (numbers) to the other side. This is accomplished by adding or subtracting the same amount to each side of the equation.
3. Isolate the variable (usually accomplished by division or multiplication).
4. Check the solution in the original equation.

Examples:

a. $4x - 5 = 11$ Add 5 to each side

$$\begin{array}{rcl} 4x - 5 &=& 11 \\ \underline{\quad +5} && \underline{+5} \\ \dfrac{4x}{4} &=& \dfrac{16}{4} \\ x &=& 4 \end{array}$$

Divide by 4

To check, substitute 4 into the original: $4(4) - 5 = 11$

$16 - 5 = 11$

$11 = 11$

b. $3x + 5 = 2x + 13$ No grouping symbols are present, no like terms to combine on either side. Start with step 2: subtract 2x from each side of the equation.

$$\begin{array}{rcl} 3x + 5 &=& 2x + 13 \\ \underline{-2x \qquad} && \underline{-2x \qquad} \\ x + 5 &=& 13 \\ \underline{\quad -5} && \underline{-5} \\ x &=& 8 \end{array}$$

Subtract 5 from each side of the equation.

To check: substitute x = 8 into the original equation:

$3x + 5 = 2x + 13$

$3(8) + 5 = 2(8) + 13$

$24 + 5 = 16 + 13$

$29 = 29$ The solution is correct because the same amount was obtained on both sides of the equation.

c. $2(x-1)+3=x-3(x+1)$ Begin by distributing and combining like terms on each side of the equation.

$2x-2+3=x-3x-3$

$$\begin{array}{rcll}
2x + 1 & = & -2x - 3 & \text{Add 2x to each side.} \\
\underline{+2x} & & \underline{+2x} & \\
4x + 1 & = & -3 & \text{Subtract 1 from each side.} \\
\underline{-1} & & \underline{-1} & \\
\frac{4x}{4} & = & \frac{-4}{4} & \text{Divide by 4} \\
x & = & -1 &
\end{array}$$

Check: $2(x-1)+3=x-3(x+1)$

$2(-1-1)+3=-1-3(-1+1)$

$2(-2)+3=-1-3(0)$

$-4+3=-1$

$-1=-1$

Practice Problems: Solve the following equations for x.

1. **$-12 = 3(2x - 8)$**
2. **$-(x+2)=2(3x-6)$**
3. **$-5(3-4x)=-6+20x-9$**

Solutions

1. **2** 2. **$\frac{10}{7}$** **3. All real numbers**

Linear Equations with Fractions

Equations are usually easier to solve if they do not contain fractions. To eliminate fractions from an equation, multiply *both* sides by the LCD (of all expressions). If you choose the correct LCD, no fractions will be present after you have multiplied.

Examples:

a. $\frac{4}{5}+n=\frac{1}{3}$ Multiply both sides (every term) by 15 (the LCD).

$15\left(\frac{4}{5}\right)+15n=15\left(\frac{1}{3}\right)$

$$\begin{array}{rcll}
12 + 15n & = & 5 & \text{Subtract 12 from each side.} \\
\underline{-12} & & \underline{-12} & \\
\frac{15n}{15} & = & \frac{-7}{15} & \text{Divide both sides by 15.}
\end{array}$$

$n = -\frac{7}{15}$

b. $2x - \frac{2x}{7} = \frac{x}{2} + \frac{17}{2}$ Multiply both sides by 14 (the LCD).

$14(2x - \frac{2x}{7}) = 14(\frac{x}{2} + \frac{17}{2})$ Note that 14 needs to be distributed on each side.

$14 \cdot 2x - 14 \cdot \frac{2x}{7} = 14 \cdot \frac{x}{2} + 14 \cdot \frac{17}{2}$

$28x - 2 \cdot \frac{2x}{1} = 7 \cdot \frac{x}{1} + 7 \cdot \frac{17}{1}$

$28x - 4x = 7x + 119$ Combine like terms

$24x = 7x + 119$ Subtract 7x from each side

$\underline{-7x \quad -7x}$

$\frac{17x}{17} = \frac{119}{17}$ Divide by 22

$x = 7$

Practice Problems: Solve the following equations:

1. $\frac{3}{5}x - \frac{2}{3} = \frac{1}{2}$
2. $3x + \frac{x}{8} = \frac{3x}{8} - \frac{1}{4}$
3. $\frac{5}{6}x + \frac{7}{3} = \frac{3}{2}x$

Solutions

1. **35/18**
2. $-\frac{1}{11}$
3. $\frac{7}{2}$

Solving Inequalities

A ***linear inequality in x*** can be written as ax + b < 0. The inequality sign might be any of the following:

$<$ "less than"
$>$ "greater than"
$\leq$ "less than or equal to"
$\geq$ "greater than or equal to"

To solve a linear inequality, we can apply any of the procedures that we have already discussed for linear equations with the following exception:

When an inequality is multiplied or divided by a negative number, the direction of the inequality sign is reversed.

Why? Consider 5 < 10. Multiply (or divide) both sides by -1.
Is -5 < -10? The relationship between the numbers has changed and the correct relationship is -5 > -10.

Examples: Solve the following inequalities. Use interval notation to express the solution set. Graph each solution set on a number line.

a. $2x + 5 < 17$ Subtract 5
$2x < 12$ Divide by 2
$x < 6$

b. $-5x \leq 30$ Divide by -5. Remember to reverse the direction of the inequality sign because you are dividing by a negative number!!!
$x \geq -6$

c. $-4(x+2) > 3x + 20$ Distribute
$-4x - 8 > 3x + 20$ Add 4x
$-8 > 7x + 20$ Subtract 20
$-28 > 7x$ Divide by 7
$-4 > x$ This should be rearranged so that "x" is read first. This inequality is equivalent to:
$x < -4$

d. $6(x-1)-(4-x) \geq 7x-8$
$6x-6-4+x \geq 7x-8$
$7x-10 \geq 7x-8$
$-10 \geq -8$ Note that after subtracting 7x from each side, the variable has dropped out. Decide whether the remaining inequality is true or false. If true, the solution set would be all real numbers. If false, there is no solution set and we write Ø.
(This is false so there is no solution.)

e. $x-6 < x+10$ Subtract x from each side.
-6 < 10 Note that again, the variable has dropped out. This time, the resulting inequality is true. Therefore, the solution set is all real numbers.

Practice Problems: Solve the following linear inequalities

1. $\boldsymbol{-3x+4 \geq 10}$
2. $\boldsymbol{-(y+2) < -2(-2y+5)}$
3. $\boldsymbol{2(x+4) \leq 2x-5}$
4. $\boldsymbol{-3(8x+3) < 4(-6x+1)}$

Solutions:

1. $\boldsymbol{x \geq 2}$
2. $\mathbf{y} \geq \boldsymbol{\frac{8}{5}}$
3. **No solution**
4. **All real numbers**

Translating Words to Algebraic Expressions

Key Words

Addition: sum, added to, total, more than, greater than, increased by
Subtraction: difference, subtracted from, less than, decreased by
Multiplication: product, times, multiplied by, twice, of
Division: quotient, divided by

Remember: Switch order for "less than" and subtracted from"

Examples

8 more than x
$x+8$

five more than twice a number
$5+2x$

7 subtracted from a number
$x-7$

twice a number subtracted from ten
$10-2x$

a number decreased by 6
$x-6$

the difference between x and 7
$x-7$

the product of eight and a number
$8x$

four times the difference of a number and 2
$4(x-2)$

30% of a number
$0.30x$

twice a number increased by eight
$2x+8$

the quotient of a number and 5
$\frac{x}{5}$

a number divided by three
$\frac{x}{3}$

one eighth of a number
$\frac{1}{8}x$

six less than m
$m-6$

three less than four times a number
$4x-3$

twice the difference of a number and four
$2(x-4)$

<u>Expressions Involving Percents</u>

The cost x, increased by 8.75% tax: $x+0.0875x$

The cost x reduced by 40% : $x-0.40x$

<u>Consecutive Integers</u>

3 consecutive integers : $x,\ x+1,\ x+2$

3 consecutive *even* integers $x,\ x+2,\ x+4$

3 consecutive *odd* integers $x,\ x+2,\ x+4$

Write each of the following as an algebraic equation:

1. Six less than a number is 12
   $x-6=12$

2. Four times the difference of a number and 9 is 27.
   $4(x-9)=27$

3. Seven more than three times a number is two times the sum of the number and five.
   $7+3x=2(x+5)$

4. The sum of a number and the number increased by 7 is 36.
   $x+(x+7)=36$

5. Six less than three times a number is one-fourth the number.

   $3x - 6 = \frac{1}{4}x$

Practice Problems: Translate the following into algebraic expressions or equations

1. **The product of a number and 8.**
2. **The difference of 6 and a number.**
3. **The sum of twice a number and 17**
4. **One sixth of a number**
5. **Four less than a number**
6. **60% of a number**
7. **Eight less than a number is fifteen**
8. **The sum of a number and the number decreased by 8 is 24**
9. **Nine less than six times a number is three times the difference of the number and seven**

Solutions

1. **8x**
2. **6 – x**
3. **2x + 17**
4. $\boldsymbol{\frac{1}{6}x}$
5. $\boldsymbol{x - 4}$
6. **0.60x**
7. **x – 8 = 15**
8. **x + (x - 8) = 24**
9. **6x – 9 = 3(x – 7)**

Solving Application Problems

Problems that are presented to us in verbal form can usually be translated into a mathematical equation that can be solved to find the solution to the original problem. Follow the strategy below to help with this process:

1. Read the problem carefully! It is sometimes helpful to underline key words that indicate a certain mathematical operation.
2. Define a variable for one of the unknown quantities. Use the statement "Let x = …."
3. If possible, write expressions for any other unknown quantities using the variable chosen in step 2.
4. Write an equation that represents the relationship between the unknown quantities.
5. Solve the equation. Use this solution to answer the question posed by the problem.
6. Check your solution according to the original wording of the problem. Note: If you check only in your equation and you have made an error in forming this equation, you may *not* notice this error.

Examples:

a. When seven times a number is decreased by 3, the result is 11. What is the number?

Let x = the number (Let x = the unknown quantity.)

$7x - 3 = 11$ (Write an equation.)

$\underline{\quad +3 \quad +3}$ (Solve)

$\frac{7x}{7} = \frac{14}{7}$

$x = 2$

The number is 2. (Answer the question)

Check: Seven times 2 decreased by 3 is 11 (14 – 3 = 11)

b. When 40% of a number is added to the number, the result is 252. What is the number?

Let x = the number (Let x = the unknown quantity.)

$0.40x + x = 252$ ***Remember, 40% of a number means 0.40 times the number

$\frac{1.40x}{1.40} = \frac{252}{1.40}$

$x = 180$

The number is 180.

Check: 40% of 180 added to 180 = 252

$0.40(180) + 840 = 252$

$72 + 180 = 252$

$252 = 252$

c. One number exceeds another by 24. The sum of the numbers is 58. What are the numbers?

Let x = the first number
x + 24 = second number ("exceeds" means is more than: addition)

Note that in this problem, there are 2 unknown quantities, therefore a second expression is used to represent the second quantity.

$$
\begin{aligned}
x + (x + 24) &= 58 \qquad \text{("Sum" indicates addition.)}\\
2x + 24 &= 58\\
\underline{\quad -24} &\underline{\;\; -24}\\
\frac{2x}{2} &= \frac{34}{2}\\
x &= 17
\end{aligned}
$$

To find the second number, replace x with 17 in the expression "x + 24".

x + 24 = 17 + 24 = 41   The numbers are {17, 41}

Check: 17 + 41 = 58

d. In 2004, the price of a sports car was approximately $80,500 with a depreciation of $7652 per year. After how many years will the car's value be $36,975?

Let x = # of years until value is $42,240

("Depreciation" means that the car is losing value each year. Be sure to multiply the loss each year (8705) by the number of years (x)).

$$
\begin{aligned}
80500 - 7652x &= 42240\\
\underline{-80500 \qquad\quad} &\underline{\;\; -80500}\\
\frac{-7652x}{-7652} &= \frac{-38260}{-7652}\\
x &= 5
\end{aligned}
$$

After 5 years the car's value will be $42240

Check: 80500 – 7652(5) = 42240
80500 – 38260 = 42240
42240 = 42240

e. After a 20% reduction, you purchase a dictionary for $25.40. What was the dictionary's price before the reduction?

Let x = price before reduction

To calculate a percentage reduction, multiply the percentage by the original price, then subtract from the original price.

$x - 0.20x = 25.40$

$\frac{0.80x}{0.80} = \frac{25.40}{0.80}$

$x = 31.75$

The original price was $31.75.

Check: 31.75(0.20) = 6.35 31.75-6.35 = $25.40

f. The selling price of a scientific calculator is $15. If the markup is 25% of the dealer's cost, what is the dealer's cost of the calculator?

Let x = dealer's cost

To calculate a percentage markup, multiply the percentage by the original price, then add the original price.

$x + 0.25x = 15$

$\frac{1.25x}{1.25} = \frac{15}{1.25}$

$x = 12$

The dealer's price is $12

Check: 12(.25) = 3 12 + 3 = 15

g. The length of a rectangular pool is 6 meters less than twice the width. If the pool's perimeter is 126 meters, what are its dimensions?

Let x = width

2x – 6 = length ***Note "less than" requires us to *subtract from*

P = 2l + 2w
126 = 2(2x – 6) + 2x
126 = 4x – 12 + 2x
126 = 6x – 12
+12 + 12
$\frac{138}{6} = \frac{6x}{6}$
23 = x

The width is 23
The length is 2(23) = 6 = 46 – 6 = 40

Check: 2(23) + 2(40) = 46 + 80 = 126

Practice problems: Solve the following problems:

1. **When eight times a number is increased by 4, the result is 36. Find the number.**

2. **When 30% of a number is subtracted from the number, the result is 35. Find the number.**

3. **Two numbers differ by eight. The sum of the numbers is 42. Find the numbers.**

4. **After a 40% discount, the price of a new cell phone is $150. Find the price of the phone before the discount.**

5. **The length of a rectangular field is six feet greater than twice the width. If the perimeter is 228ft, find the length and width of the field.**

6. **A new pair of sneakers cost $75.45 after tax (rate of 8.75%) has been added. Find the price of the sneakers before tax. (Round your answer to the nearest cent.)**

Solutions

1. **4 2. 50 3. 17 and 25 4. $250 5. L = 78ft; w = 36 ft 6. $69.38**

Rules of Exponents

The Product Rule: $b^m \cdot b^n = b^{m+n}$
When multiplying exponential expressions with the *same base*, add the exponents.

Examples:

a. $x^{11} \cdot x^5 = x^{16}$ b. $x^{-6} \cdot x^{12} = x^6$

c. $2^{-3} \cdot 2 = 2^{-2}$ (remember: 2 really has an exponent of 1 that is added to the exponent -3)

d. $(11x^5)(9x^{12}) = 99x^{17}$ e. $(-5x^4y)(-6x^7y^{11}) = 30x^{11}y^{12}$

The Quotient Rule: $\frac{b^m}{b^n} = b^{m-n}$ $b \neq 0$

When dividing exponential expressions with the *same base*, subtract the exponents.

Examples:

a. $\frac{3^8}{3^4} = 3^4$ b. $\frac{x^{30}}{x^{10}} = x^{20}$ c. $\frac{35a^{14}b^7}{-7a^7b^4} = -5a^7b^3$

The Zero- Exponent Rule: If b is any real number other than 0, $b^0 = 1$

Examples:

a. $(-9)^0 = 1$ b. $(3x)^0 = 1$

c. $3x^0 = 3 \cdot 1 = 3$ (note: the exponent applies only to the x, NOT to the 3)

d. $-9^0 = -1$ (note: the exponent applies only to the 9, not to the “-“.)

The Negative Exponent Rule

If b is any real number other than 0 and n is a natural number, then $b^{-n} = \frac{1}{b^n}$

Examples:

a. $2^{-6} = \frac{1}{2^6} = \frac{1}{64}$ b. $2^{-3} \cdot 2 = 2^{-2} = \frac{1}{2^2} = \frac{1}{4}$

c. $(-3)^{-2} = \frac{1}{(-3)^2} = \frac{1}{9}$

The Power Rule: $(b^m)^n = b^{mn}$

When an exponential expression is raised to a power, multiply the exponents.

Examples:

a. $(3^3)^3 = 3^9$ b. $(x^{11})^5 = x^{55}$ c. $(x^{-6})^4 = x^{-24} = \dfrac{1}{x^{24}}$

Products to Powers: $(ab)^n = a^n b^n$

When a product is raised to a power, raise each factor to that power.

Examples:

a. $(6x^4)^2 = 6^2(x^4)^2 = 36x^8$ b. $(-3x^4y^6)^3 = (-3)^3(x^4)^3(y^6)^3 = -27x^{12}y^{18}$

Quotients to Powers

If b is a nonzero real number, then $(\dfrac{a}{b})^n = \dfrac{a^n}{b^n}$

When a quotient is raised to a power, raise the numerator *and* denominator to that power.

Examples:

a. $(-\dfrac{6}{y})^3 = \dfrac{(-6)^3}{y^3} = -\dfrac{216}{y^3}$ b. $(\dfrac{x^2}{2})^4 = \dfrac{(x^2)^4}{2^4} = \dfrac{x^8}{16}$

Simplifying Exponential Expressions

An expression is simplified when:

1. No parenthesis appear
2. No powers are raised to powers
3. Each base occurs only once
4. No negative exponents or 0 exponents appear.

Examples:

a. $\dfrac{35a^{14}b^6}{-7a^7b^3} = \dfrac{-5a^7b^3}{1} = -5a^7b^3$ (Divide the coefficients, subtract exponents.)

b. $(10x^2)^{-3} = \dfrac{1}{(10x^2)^3} = \dfrac{1}{1000x^6}$ (Eliminate the negative exponent, raise each factor to the third power in the denominator.)

Practice Problems
Simplify the following:

1. $\frac{4x^8y^{-2}}{8x^7y^3}$ **2.** $(2x^2)^3(3xy^4)$ **3.** $(5x^{-4}y^{-1})(x^7y^0)$ **4.** $(5y^3)^{-2}$ **5.** $(2x^5y^4)^0$

Solutions

1. $\frac{x}{2y^5}$ **2.** $24x^7y^7$ **3.** $\frac{5x^3}{y}$ **4.** $\frac{1}{25y^6}$ **5. 1**

Multiplying Polynomials

(Monomial)(Monomial): Multiply coefficients, add exponents of like bases.

Examples:

a. $(-6x^5)(3x^8) = -18x^{13}$ b. $(-2y^8)(-6y^{10}) = 12y^{18}$

(Binomial)(Binomial)

We can use the acronym FOIL to describe how to multiply two binomials:

Ex. (x + 6)(x – 5)

F stands for the first terms in each binomial: $x \cdot x = x^2$
O stands for the outer terms: $x \cdot -5 = -5x$
I stands for the inner terms: $6 \cdot x = 6x$
L stands for the last terms in each binomial: $6 \cdot -5 = -30$

Combine like terms and write your answer in standard form: $x^2 - 5x + 6x - 30 = x^2 + x - 30$

Examples:

a. (3x – 5)(7x + 2)
$(3x)(7x) + (3x)(2) + (-5)(7x) + (-5)(2)$
$21x^2 + 6x - 35x - 10$
$21x^2 - 29x - 10$

Practice combining the outer and inner terms (if like) mentally and just writing the final answer. It will be important to be able to perform this multiplication quickly as you progress through mathematics courses.

b. $(7x^2 - 2)(3x^2 - 5)$
$21x^4 - 35x^2 - 6x^2 + 10$
$21x^4 - 41x^2 + 10$

c. $(7x^3+5)(x^2-2)$

$7x^5-14x^3+5x^2-10$ Note: there are no like terms to combine here.

Practice Problems

Multiply the following:

1. (x – 6)(x + 8) 2. (2x – 9)(3x + 5) 3. $(\mathbf{3y^2-5})(\mathbf{2y^2+7})$ **4.** $(\mathbf{2x+7})(\mathbf{4y-5})$

Solutions:

1. $\mathbf{x^2+2x-48}$ **2.** $\mathbf{6x^2-17x-45}$ **3.** $\mathbf{6y^4+11y^2-35}$

4. $\mathbf{8xy-10x+28y-35}$

Factoring Polynomials

Factoring is the process of breaking a polynomial into a product. It is the reverse of multiplying.

Common Factors

Step 1 of factoring is always to "remove" the GCF (greatest common factor). "Remove" means *divide each term by the GCF.*

Ex. $16x-24$ The GCF of 16 x and -24 is 8. Write the GCF outside a set of parenthesis. Mentally divide each term by 8. Write each resulting quotient inside the parenthesis.

Ans: $8(2x-3)$

Note: This answer can be checked by distributing. When the 4 is distributed, the result should match the original problem. If it does not, the factorization is not correct.

Ex. $6x^4-18x^3+12x^2$ The GCF is $6x^2$. Note that when variables are involved, use the smallest exponent of each common variable as part of the GCF.

$6x^2(x^2-3x+2)$ (Remember to subtract exponents when dividing like bases).

Practice Problems

Factor each of the following:

1. 24x – 12 2. $\mathbf{8y^2-4y}$ **3.** $\mathbf{4x^3-10x^2+2x}$

Solutions

1. **12(2x – 1)** **2.** $\mathbf{4y(2y-1)}$ **3.** $\mathbf{2x(2x^2-5x+1)}$

Factoring Trinomials of the Form ax^2+bx+c **with a = 1**

Remember, factoring is the reverse of multiplying.
Examine these multiplication problems:

a.(x – 3)(x – 5) b.(x + 6)(x + 1) c. (x + 4)(x – 3) d. (x + 5)(x – 2)

$x^2-8x+15$ x^2+7x+6 x^2+x-12 $x^2+3x-10$

Our goal is to get from these answers back to the factored form.

Use the **Trial and Error** method, incorporating some guidelines.

Ex. x^2+7x+6 (a = 1, b = 7, c = 6)
Form of answer:
(x +)(x +) We must fill in the blanks with factors whose product is c

Factors of 6 (c): $6\cdot1$ $3\cdot2$

Which pair adds up to 7 (b): $6\cdot1$

(x + 6)(x + 1) Check by multiplying (FOIL)

Examples:

a. $x^2+8x+15$ (a = 1, b = 8, c = 15)
Form of answer:
(x +)(x +) We must fill in the blanks with factors whose product is c

Factors of 15 (c): $15\cdot1$ $3\cdot5$

Which pair adds up to 8 (b): $3\cdot5$

(x + 5)(x + 3) Check by multiplying (FOIL)

Note that (x + 3)(x + 5) is also a correct answer. The order of the factors in the answer does not matter.

b. $x^2-14x+45$ Factors of 45 that add up to -14 are -9 and -5
(x – 9)(x – 5)

c. $x^2 - 4x - 5$ Factors of -5 that add up to -4 are -5 and +1
(x – 5)(x + 1)

d. $x^2 + 3x - 10$ Factors of -10 that add up to +3 are +5 and -2
(x + 5)(x – 2)

Note: All of these answers can be checked by FOILing!!!
******Notice that when the sign of c (the constant) is +, the signs in each factor of your answer will be the same; the sign of b (the middle term) will tell you if both are – or +.**
******Notice that when the sign of c (the constant) is -, the signs in each factor of your answer will be different; the sign of b (the middle term) will tell you the sign of the larger number.**

Practice Problems

1. $\boldsymbol{x^2 + 11x + 18}$ **2.** $\boldsymbol{y^2 + y - 20}$ **3.** $\boldsymbol{y^2 - 15y + 56}$ **4.** $\boldsymbol{x^2 - 17x - 30}$

5. $\boldsymbol{y^2 - 2y - 8}$

Solutions:

1. (x + 2)(x + 9) 2. (y + 5) (y – 4) 3. (y – 8)(y – 7) 4. Prime (cannot be factored)

5.(y – 4)(y + 2)

Factoring Trinomials of the form $ax^2 + bx + c$ **with a ≠ 1**

Use trial and error but there will be more possibilities to consider.

Ex. $2x^2 + 5x + 3$

The only way to break up $2x^2$ is 2x · x. The only way to break up 3 is 3·1.

Our possibilities are:

(2x + 3)(x + 1) OR (2x + 1)(x + 3)

A quick check of the inners and outers (from FOIL) will determine the correct factorization. They must produce 5x when combined.

(2x + 3)(x + 1) OR	(2x + 1)(x + 3)
inners: 3x, outers: 2x	inners: x, outers: 6x
combined: 5x	combined: 7x

The first factorization is correct!!

Examples:

a. $6x^2 - 17x + 12$

The possible factors of $6x^2$ are $6x \cdot x$ and $3x \cdot 2x$. The possible factors of 12 are $12 \cdot 1$, $6 \cdot 2$ and $4 \cdot 3$ (or each pair with both negative factors). The sign of b (-17) is negative indicating that the factors of 12 will both be negative.

Trial 1: (3x – 3)(2x – 4) outers: -12x inners: -6x combined: -18x (incorrect)

Trial 2: (3x – 4)(2x – 3) outers: -9x. inners: -8x combined: -17x (correct)

b. $9x^2 + 5x - 4$

The possible factors of $9x^2$ are $9x \cdot x$ and $3x \cdot 3x$. The possible factors of -4 are $-4 \cdot 1$, $4 \cdot -1$, and $2 \cdot -2$.

Trial 1: (3x – 4)(3x + 1) outers: 3x inners: -12x combined: -9x (incorrect)

Trial 2: (9x + 1)(x – 4) outers -36x inners: x combined: -35x (incorrect)
Note: After checking the outers, you might quickly decide that it appears too large to continue and go on to the next trial.

Trial 3: (9x - 4)(x + 1) outers: 9x inners: -4x combined: 5x (correct)

Practice Problems:

1. $\mathbf{2x^2 - x - 15}$ **2.** $\mathbf{5y^2 - 32y + 12}$ **3.** $\mathbf{3x^2 + 10x + 7}$

Solutions:

1. **(2x + 5)((x – 3) 2. (5y – 2)(y – 6) 3. (3x + 2)(x + 4)**

Factoring the Difference of Two Squares

When factoring an expression containing 2 terms separated by a "-" sign, each of which is a perfect square, the factorization will always be of the following form:

$A^2 - B^2 = (A + B)(A - B)$

Note: the factors in your answer can be in either order. It doesn't matter if the "+" is first or the "-" is first.

Examples:

a. $x^2 - 144$ A variable expression containing an even exponent is a perfect square, 144 is also a perfect square.
(x + 12)(x – 12)

b. $36x^2 - 49y^2$ = (6x + 7y)(6x – 7y)

Practice Problems

1. $\boldsymbol{x^2 - 25}$ **2.** $\boldsymbol{4y^2 - 81x^2}$ **3.** $\boldsymbol{100 - 144x^2}$

Solutions

1. **(x – 5)(x + 5) 2. (2y – 9x)(2y + 9x) 3. (10 - 12x)(10 + 12x)**

Repeated Factorizations

In order to "completely factor" an expression, it must be broken down as far as possible. This often requires more than one factoring step. ***Always remove the GCF first if possible***.

Examples:

a. $81x^4 - 1$ First, recognize the expression as the difference of 2 squares
$(9x^2 + 1)(9x^2 - 1)$ Note that the second factor is again the difference of 2 squares and should be factored again.
$(9x^2 + 1)(3x + 1)(3x - 1)$

b. $50x^2 - 10x - 12$ First, remove (divide by) the GCF of 2
$2(25x^2 - 5x - 6)$ Continue factoring by trial and error
$2(5x + 2)(5x - 3)$

Practice Problems

1. $\boldsymbol{81x^2 - 9y^2}$ **2.** $\boldsymbol{3x^2 - 48}$ **3.** $\boldsymbol{12x^2 + 4x - 16}$

Solutions:

1. **(9(3x + y)(3x – y) 2. 3(x + 4)(x – 4) 3. 4(3x + 4)(x – 1)**

Rational Expressions

Simplifying a Rational Expression

To simplify a rational expression:

1. Factor the numerator and denominator completely
2. Divide both the numerator and denominator by any common factors.
3. State any numbers that are excluded from the domain of the original expression.

If the expression is simplified, the numerator and denominator will not contain any common factors other than 1.

Examples: Simplify.

a. $\frac{4x-8}{x^2-4x+4} = \frac{4(x-2)}{(x-2)(x-2)}$ Factor the top and bottom

$= \frac{4}{x-2} \qquad x \neq 2$

The top and bottom were divided by the common factor: (x-2). The restrictions on the domain were stated.

b. $\frac{y^2-4y-5}{y^2+5y+4} = \frac{(y-5)(y+1)}{(y+4)(y+1)} = \frac{y-5}{y+4} \qquad y \neq -4,-1$ (Note: the restrictions include ALL values that make the *original* expression undefined, not only those that make the simplified version undefined.

Practice Problems:

Simplify the following expressions:

1. $\frac{x^2+2x-8}{x-2}$ **2.** $\frac{y^2-7y+10}{y^2-3y-10}$ **3.** $\frac{x-2}{4x^2-13x+10}$

Solutions:

1. **x + 4** **2.** $\frac{y-2}{y+2}$ **3.** $\frac{1}{4x-5}$

Multiplication and Division of Rational Expressions

Multiplying Rational Expressions

1. Factor all numerators and denominators completely.
2. Divide numerators and denominators by common factors (you can divide one factor anywhere on the top by the identical factor anywhere on the bottom.)
3. Multiply the remaining factors in the numerator; multiply the remaining factors in the denominator.

Examples:

a. $\frac{x^2-4}{x^2-4x+4}\cdot\frac{2x-4}{x+2}$ Factor

$\frac{(x+2)(x-2)}{(x-2)(x-2)}\cdot\frac{2(x-2)}{x+2}$ Divide the common factors (x – 2), (x – 2) and (x + 2)

$\frac{2}{1}=2$ $x\neq 2,-2$

b. $\frac{x^2+6x+9}{x^3+27}\cdot\frac{1}{x+3}$

$\frac{(x+3)(x+3)}{(x+3)(x^2-3x+9)}\cdot\frac{1}{x+3}$

$\frac{1}{(x^2-3x+9)}$ $x\neq -3$

Dividing Rational Expressions

To divide rational expressions, multiply the first expression by the reciprocal of the second expression (as we do with rational numbers). Remember, the reciprocal of an expression is formed by inverting (flipping it upside down). Follow the steps for multiplication of rational expressions. Exclude from the domain, any value(s) that makes the original expressions undefined, as well as any value(s) that makes the inverted expression undefined.

Examples:

a. $\dfrac{x^2-4}{x-2} \div \dfrac{x+2}{4x-8}$ Keep the first expression, multiply by the reciprocal of the second.

$\dfrac{x^2-4}{x-2} \cdot \dfrac{4x-8}{x+2}$ Factor

$\dfrac{(x-2)(x+2)}{x-2} \cdot \dfrac{4(x-2)}{x+2}$ Divide the common factors

$4(x-2) \qquad x \neq 2,-2$

b. $\dfrac{x^2+x}{x^2-4} \div \dfrac{x^2-1}{x^2+5x+6}$ Keep the first expression, multiply by the reciprocal of the second.

$\dfrac{x^2+x}{x^2-4} \cdot \dfrac{x^2+5x+6}{x^2-1}$ Factor

$\dfrac{x(x+1)}{(x-2)(x+2)} \cdot \dfrac{(x+3)(x+2)}{(x+1)(x-1)}$ Divide the common factors

$\dfrac{x}{x-2} \cdot \dfrac{(x+3)}{(x-1)}$ Multiply numerators and denominators.

$\dfrac{x(x+3)}{(x-2)(x-1)} \qquad x \neq 2,-2,-3,1,-1$

Practice Problems: Perform the indicated operation and simplify.

1. $\dfrac{x^2+7x+12}{6x} \cdot \dfrac{x^2-4x}{x^2-x-12}$ **2.** $\dfrac{(x+3)^2}{5x^2} \cdot \dfrac{10x}{x^2-9}$ **3.** $\dfrac{x^2-16}{3x^2-12x} \div \dfrac{5x^2+20x}{3x^2+6x}$ **4.** $\dfrac{x^5y^8}{2z} \div \dfrac{4xy}{3z}$

Solutions:

1. $\dfrac{x+4}{6}$ **2.** $\dfrac{2(x+3)}{x(x-3)}$ **3.** $\dfrac{x+2}{5x}$ **4.** $\dfrac{3x^5y^7}{8}$

Addition and Subtraction of Rational Expressions

Same Denominator

Add or subtract the numerators, the denominator remains the same; simplify.

Examples:

a. $\frac{x^2-4x}{x^2-x-6}+\frac{4x-4}{x^2-x-6}$ Add the numerators, denominator remains the same

$\frac{x^2-4x+4x-4}{x^2-x-6}=\frac{x^2-4}{x^2-x-6}$ Factor to simplify.

$\frac{(x+2)(x-2)}{(x-3)(x+2)}$ Divide the like factors.

$\frac{x-2}{x-3}$ $x \neq 3,-2$

b. $\frac{2x+3}{3x-6}-\frac{3-x}{3x-6}$ Subtract numerators (remember to distribute the "-" sign).

$\frac{2x+3-(3-x)}{3x-6}=\frac{2x+3-3+x}{3x-6}=\frac{3x}{3x-6}$ Factor to simplify

$\frac{3x}{3(x-2)}=\frac{x}{x-2}$ $x \neq 2$

Different Denominators

The expressions must be rewritten as equivalent expressions containing the least common denominator (LCD). To find the LCD:

1. Factor each denominator completely.
2. List the factors of the first denominator.
3. Add to the list, any factors of the second denominator that do not appear in the list (there is no need to repeat any common factors).
4. The product of all the factors in the list is the LCD of the expressions.

Examples:

a. Find the LCD of $\frac{3x}{x^2+3x-10}$ *and* $\frac{2x}{x^2+x-6}$ Factor each denominator completely.

$$\frac{3x}{(x+5)(x-2)} \qquad \frac{2x}{(x+3)(x-2)}$$

Factors of first denominator: (x + 5) (x – 2). From the second denominator, add (x + 3) to the list. So the LCD is (x + 5)(x – 2)(x + 3). Note: these factors can appear in any order but all three must appear in the LCD.

b. Find the LCD of $\frac{3}{5x+2}$ *and* $\frac{5x}{25x^2-4}$ Factor each denominator completely.

$$\frac{3}{5x+2} \qquad \frac{5x}{(5x+2)(5x-2)}$$

Factors of first denominator: (5x + 2). From the second denominator, add (5x – 2) to the list. (There is no need to repeat the identical factor.) So the LCD is (5x + 2)(5x – 2). Note: these factors can appear in any order.

To add or subtract rational expressions that have different denominators:

1. Find the LCD of the rational expressions.
2. Rewrite each expression as an equivalent one using the LCD. (Multiply the numerator and denominator by any factor needed to change the denominator into the LCD.)
3. Add or subtract the numerators; keep the denominator the same.
4. Simplify if possible.

Examples:

a. $\frac{8}{x-2}+\frac{2}{x+3}$ LCD is (x – 2)(x + 3). To find the equivalent expressions, multiply the first fraction by (x + 3) and the second fraction by (x – 2). Remember, multiply BOTH the numerator and denominator.

$\frac{8(x+3)}{(x-2)(x+3)}+\frac{2(x-2)}{(x-2)(x+3)}$ Add the numerators (remember to distribute!)

$\frac{8(x+3)+2(x-2)}{(x-2)(x+3)}=\frac{8x+24+2x-4}{(x-2)(x+3)}=\frac{10x+20}{(x-2)(x+3)}$ Factor to see if the expression can be simplified.

$\frac{10(x+2)}{(x-2)(x+3)}$ $x \neq 2,-3$ There are no common factors to divide.

b. $\frac{x}{x^2-2x-24}-\frac{x}{x^2-7x+6}$ Factor to find the LCD.

$\frac{x}{(x-6)(x+4)}-\frac{x}{(x-6)(x-1)}$ LCD: (x – 6)(x + 4)(x – 1) To find the equivalent expressions, multiply the first fraction by (x – 1) and the second fraction by (x + 4). Remember, multiply BOTH the numerator and denominator.

$\frac{x(x-1)}{(x-6)(x+4)(x-1)}-\frac{x(x+4)}{(x-6)(x+4)(x-1)}$ Subtract the numerators (remember to distribute!)

$\frac{x(x-1)-x(x+4)}{(x-6)(x+4)(x-1)}=\frac{x^2-x-x^2-4x}{(x-6)(x+4)(x-1)}=\frac{-5x}{(x-6)(x+4)(x-1)}$ $x \neq 5,-4,1$

Practice Problems: Perform the indicated operation and simplify.

1. $\frac{5x-4}{x+8}+\frac{44}{x+8}$ **2.** $\frac{7}{x+4}+\frac{2}{x}$ **3.** $\frac{x+2}{x^2-x-6}+\frac{x-3}{x^2-8x+15}$ **4.** $\frac{5x}{4x-8}-\frac{x}{x+2}$ **5.** $\frac{3}{x+2}-\frac{5}{x-7}$

Solutions:

1. 5 **2.** $\frac{9x+8}{x(x+4)}$ **3.** $\frac{2x-8}{(x-3)(x-5)}$ **4.** $\frac{x^2+18x}{4(x-2)(x+2)}$ **5.** $\frac{-2x-31}{(x+2)(x-7)}$

Solving Rational Equations

A ***rational equation*** contains one or more rational expressions (remember, these contain variables in the denominator). Follow the same procedure for solving as above with linear equations containing fractions. Sometimes, one or more denominators will need to be factored in order to determine the LCD. Remember to avoid any values that make any rational expression involved undefined (values that make the denominator = 0).

Example:

$\frac{5}{2x}-\frac{8}{9}=\frac{1}{18}-\frac{1}{3x}$ LCD: 18x (all denominators divide evenly into 18x). Multiply both sides by 18x.

$18x(\frac{5}{2x}-\frac{8}{9})=18x(\frac{1}{18}-\frac{1}{3x})$ Note that 18x needs to be distributed on each side.

$$18x\cdot\frac{5}{2x}-18x\cdot\frac{8}{9}=18x\cdot\frac{1}{18}-18x\cdot\frac{1}{3x}$$

$$9\cdot\frac{5}{1}-2x\cdot\frac{8}{1}=x\cdot\frac{1}{1}-6\cdot\frac{1}{1}$$

$45-16x=x-6$ Add 16x to each side.

$$\begin{array}{rcl} 45-16x &=& x-6 \\ +16x & & +16x \\ \hline 45 &=& 17x-6 \end{array}$$

Add 6 to each side

$$\begin{array}{rcl} 45 &=& 17x-6 \\ +6 & & +6 \\ \hline 51 &=& 17x \end{array}$$

Divide by 17

$$\frac{51}{17}=\frac{17x}{17}$$

$$3 = x$$

Check: $\frac{5}{2x}-\frac{8}{9}=\frac{1}{18}-\frac{1}{3x}$

$$\frac{5}{2\cdot 3}-\frac{8}{9}=\frac{1}{18}-\frac{1}{3\cdot 3}$$

$$\frac{5}{6}-\frac{8}{9}=\frac{1}{18}-\frac{1}{9}$$

$$\frac{15}{18}-\frac{16}{18}=\frac{1}{18}-\frac{2}{18}$$

$$-\frac{1}{18}=\frac{-1}{18}$$

Practice Problems: Solve the following equations.

1. $\frac{10}{x}+\frac{3}{2}=\frac{x}{10}$ **2.** $\frac{x}{6}+\frac{2}{3}=\frac{x}{5}-\frac{1}{3}$ **3.** $\frac{x}{4}+\frac{x}{3}=\frac{7}{6}$

Solutions:

1. 20, -5 **2. 30** **3. 2**

Complex Fractions (Complex Rational Expressions)

Simplifying

To simplify a complex rational expression:

1. Find the LCD of all the rational expressions in the numerator and denominator.
2. Multiply *each* term of the expression by the LCD (this should eliminate all of the rational expressions).
3. Simplify the resulting expression by factoring (if possible) and dividing out common factors.

Examples:

a. $\dfrac{\frac{x}{4}-1}{x-4}$ The LCD is 4. Multiply each term by 4.

$$\frac{\frac{x}{4}\cdot 4-1\cdot 4}{x\cdot 4-4\cdot 4}=\frac{x-4}{4x-16}$$ Factor

$$\frac{x-4}{4(x-4)}=\frac{1}{4}$$ $x \neq 4$

b. $\dfrac{\frac{1}{x}+\frac{1}{y}}{xy}$ The LCD is xy. Multiply each term by xy.

$$\frac{\frac{1}{x}\cdot xy+\frac{1}{y}\cdot xy}{xy\cdot xy}=\frac{y+x}{x^2y^2}$$ $x \neq 0, y \neq 0$ (No factoring is possible.)

Practice Problems: Simplify the following rational expressions:

1. $\dfrac{\frac{1}{x}+3}{\frac{1}{x}+\frac{3}{x}}$ **2.** $\dfrac{\frac{3}{x}+\frac{2}{x^2}}{5-\frac{1}{x}}$ **3.** $\dfrac{\frac{1}{x}-\frac{1}{y}}{\frac{1}{x}+\frac{1}{y}}$

Solutions:

1. $\dfrac{3x+1}{4}$ **2.** $\dfrac{3x+2}{x(5x-1)}$ **3.** $\dfrac{y-x}{y+x}$

Solving Systems of Linear Equations

Substitution Method

Steps:
1. Solve one of the equations (either one) for one variable (either one).
2. Substitute the expression obtained in Step 1 into the *other* equation. (only one variable should now be present.)
3. Solve this resulting equation.
4. Re-substitute the answer obtained in Step 3 into the either original equation. Solve for the other variable.
5. Check the solution in BOTH original equations.

Examples: Solve the following systems of equations by substitution.

a. 2x – 3y = -13 (1) (The equations are numbered so that they
y = 2x + 7 (2) can be easily referred to.)

Equation (2) is already solved for y so step 1 can be omitted. The expression 2x + 7 will be substituted into Equ. (1) in place of y:
2x – 3y = -13
2x – 3(2x + 7) = -13 Distribute and solve for x.
2x – 6x – 21 = -13
-4x – 21 = -13
-4x = 8
x = -2

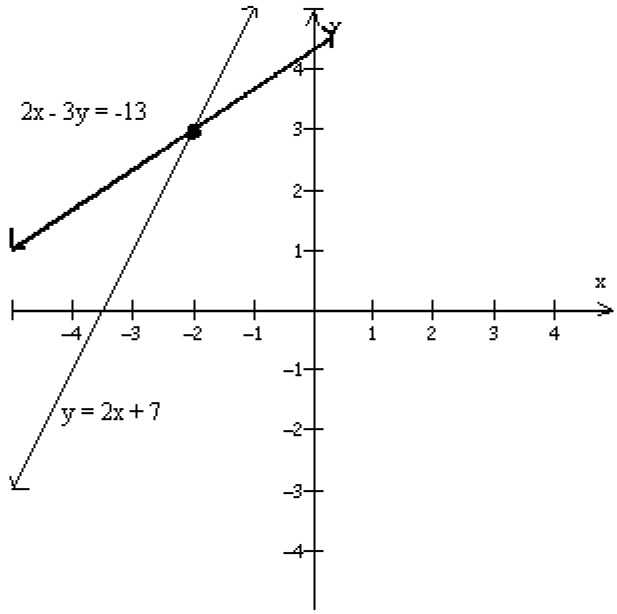
Re-substitute into equ (2) (or equ (1) – I just chose equ (2)) to find y:
y = 2x + 7
y = 2(-2) + 7
y = -4 + 7
y = 3

Solution (-2, 3)

Check: $2x – 3y = -13$ | $y = 2x + 7$

Check: $2x - 3y = -13$	$y = 2x + 7$
$2(-2) - 3(3) = -13$	$3 = 2(-2) + 7$
$-4 - 9 = -13$	$3 = -4 + 7$
$-13 = -13$ true	$3 = 3$ true

Note: When both lines are graphed, the point of intersection is (-2, 3) as can be seen on the graph below:



b. $2x + 5y = 1$ (1)
$-x + 6y = 8$ (2)

It looks like it will be easiest to solve equ. (2) for x (the coefficient is -1 which should not create any fractions when solving.)

$-x + 6y = 8$

$-x = 8 – 6y$ Divide all terms by -1

$x = -8 + 6y$

$x = 6y -8$

Substitute into equ. (1):

$2x + 5y = 1$

$2(6y – 8) + 5y = 1$

$12y – 16 + 5y = 1$

$17y – 16 = 1$

$17y = 17$

$y = 1$

Re-substitute into equ. (2) (or equ (1). Note: As we have already isolated x in our first step, we could also re-substitute into that expression instead (x = 6y – 8) as long as we are sure that no mistakes have been made in isolating x.

-x + 6y = 8
-x + 6(1) = 8
-x + 6 = 8
-x = 2
x = -2

Solution: (-2, 1)

Check: 2x + 5y = 1 -x + 6y = 8
2(-2) + 5(1) = 1 -(-2) + 6(1) = 8
-4 + 5 = 1 2 + 6 = 8
1 = 1 true 8 = 8 true

c. $y = -\frac{1}{2}x + 2$ (1)

$y = \frac{3}{4}x + 7$ (2)

Both equations have one variable already solved for. Let's substitute equ. (2) into equ. (1):

$y = -\frac{1}{2}x + 2$

$\frac{3}{4}x + 7 = -\frac{1}{2}x + 2$ Multiply all terms by 4 to clear fractions.

3x + 28 = -2x + 8 Solve for x
5x = -20
x = -4 Re-substitute into equ. (2) to find y.

$y = \frac{3}{4}x + 7$

$y = \frac{3}{4}(-4) + 7$

y = -3 + 7
y = 4

Solution (-4, 4)

Check: $y = -\frac{1}{2}x + 2$ $y = \frac{3}{4}x + 7$

$4 = -\frac{1}{2}(-4) + 2$ $4 = \frac{3}{4}(-4) + 7$

$4 = 2 + 2$ $4 = -3 + 7$
4 = 4 true 4 = 4 true

Practice Problems
Solve the following systems by substitution:

1. **$y = 2x - 3$**   **2. $2x - y = 7$**   **3. $4x - y = 1$**
   $3x - 5y = 1$ **$x + 2y = 6$**   **$10x + \frac{1}{2}y = 1$**

Solutions:

1. **(2, 1)**   **2. (4, 1)**   **3. (1/8, -1/2)**

Addition Method

Some textbooks refer to this method as solving by "elimination", meaning that one variable is eliminated (usually by addition). In this method, the like terms in each equation are lined up, added (so that one variable is eliminated) and the resulting equation is solved. This answer is then substituted into either original equation to find the other value. An example of the simplest type is provided below with more detailed steps to follow:

Ex. Solve by addition:
$x + y = 6$
$x - y = -2$ Add the equations together (notice that the "ys" drops out).

$x + y = 6$
$\underline{x - y = -2}$
$2x = 4$ Solve for x
$x = 2$

Substitute into either original to find y:
$x + y = 6$
$2 + y = 6$
$y = -4$

The solution is (2, -4)

It can be verified that this solution is correct when it is substituted into BOTH of the original equations.

Sometimes adding the original equations will not result in one of the variables being eliminated. We can adjust one (or both) equations to force this to happen. To do this, multiply one (or both) by nonzero constants so that the coefficients of one of the variables are opposites. Then, when the equations are added, that variable will be eliminated.

Ex. $2x + 5y = 1$ (1)
$-x + 6y = 8$ (2)

If equ (2) is multiplied by 2 (every term), the system becomes:

$$\begin{array}{rl} 2x + 5y = 1 & \\ \underline{-2x + 12y = 16} & \text{Add the resulting equations:} \\ 17y = 17 & \\ y = 1 & \text{Substitute into either original equation (I'll use equ 1)} \end{array}$$

$2x + 5(1) = 1$
$2x + 5 = 1$
$2x = -4$
$x = -2$

Solution: $(-2, 1)$

Check this answer in BOTH original equations to be sure that it is correct.

Summary of Steps:

1. Rewrite (if necessary) each equation in the form $Ax + By = C$ (this is so like terms, and the "=" signs are lined up).
2. If necessary, multiply one or both equations by a nonzero constant (so that the coefficients of one of the variables are opposites).
3. Add the resulting equations.
4. Solve for the remaining variable.
5. Substitute into either of the original equations to find the other variable.
6. Check the solutions in BOTH of the original equations.

Examples: Solve the following equations using the addition method.
a. $3x – 7y = 13$ (1)
$6x + 5y = 7$ (2)

In this case, we can multiply equ (1) by -2 to make the coefficients of x opposites. To indicate this in your work, write the following:
$-2(3x – 7y = 13)$
$6x + 5y = 7$

$$\begin{array}{rl} -6x + 14y = -26 & \\ \underline{6x + 5y = 7} & \text{Add the equations} \\ 19y = -19 & \\ y = -1 & \text{Substitute into equ (1) (or equ (2))} \end{array}$$

$3x - 7(-1) = 13$ equ (1)
$3x + 7 = 13$
$3x = 6$
$x = 2$

Solution: (2, -1)

Check:	$3x - 7y = 13$	$6x + 5y = 7$
	$3(2) - 7(-1) = 13$	$6(2) + 5(-1) = 7$
	$6 + 7 = 13$	$12 - 5 = 7$
	$13 = 13$	$7 = 7$

a. $2x + 3y = -16$
   $5x - 1\text{-}y = 30$ (Multiply the top equation by -5 and the bottom equation by 2)

   $-10x - 15y = 80$
   $\underline{10x - 20y = 60}$ (Add)
   $-35y = 140$ (Divide by -35)
   $y = -4$
   Substitute into either original equation (I used the top one):
   $2x + 3(14) = -16$
   $2x - 12 = -16$
   $2x = -4$
   $x = -2$

   The solution is (-2, -4). Check this in both original equations.

Practice Problems:
Solve the following systems of equations by the Addition method

1. $x - y = -4$	**2. $4x - 3y = 2$**	**3. $3x + 1 = -5y$**
$-x+6y = -6$	**$2x + 5y = 14$**	**$16 - 8 = 7x$**

Solutions:

1. (-6, -2)	**2. (2, 2)**	**3. (8, -5)**

Linear Systems with No Solution or Infinitely Many Solutions

It is possible that a linear system has ***no solution*** (no ordered pair satisfies BOTH equations at the same time) or ***infinitely many solutions*** (there is an endless list of ordered pairs that satisfy both equations). If the equations were graphed on the same set of axes, a system with no solution would be two parallel lines. A system with infinitely many solutions would be the same identical line and any point on that line is a solution.

Ex. Solve the following system by addition:
$6x + 2y = 7$
$y = 2 – 3x$ Line up in $Ax + By = C$ form:

$6x + 2y = 7$ (1)
$3x + y = 2$ (2) Multiply equ (2) by -2

$6x + 2y = 7$
$\underline{-6x – 2y = -4}$ Add
$0 + 0 = 3$
$0 = 3$
Both variables were eliminated and the resulting equation is False. Therefore, this system has no solution. We can use the symbol for "empty set" to indicate this: Ø

Ex. Solve the following system by addition:
$4x – 2y = 2$ (1)
$2x – y = 1$ (2) Multiply equ (2) by-2:

$4x – 2y = 2$
$\underline{-4x + 2y = -2}$ Add
$0 + 0 = 0$
Both variables were eliminated and the resulting equation is True. Therefore, this system has an infinite number of solutions. Set builder notation (using either original equation) is often used to indicate all of the solutions:

$\{(x, y)| 4x – 2y = 2\}$ OR $\{(x, y)| 2x – y = 1\}$

These are read "the set of all ordered pairs (x, y) such that $4x – 2y = 2$"
OR "the set of all ordered pairs (x, y) such that $2x – y = 1$" meaning that the equation in the solution must be satisfied by the ordered pair in order to be a solution.

Practice Problems
Solve the following systems by either method

1. $2x – 4y = 7$
   $4x + 8y = -14$
2. $2x + y = 3$
   $-4x – 2y = 5$

Solutions:

1. **Infinite number of solutions** **2. No solution**

Radicals

Product Rule for Square Roots (a, b are nonnegative real numbers)

$\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$ and $\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$

The square root of a product is the product of the square roots.

Examples: a. $\sqrt{36} = \sqrt{9} \cdot \sqrt{4} = 3 \cdot 2 = 6$ b. $\sqrt{25} \cdot \sqrt{4} = \sqrt{100} = 10$

We will use this property to simplify square roots.

Simplifying Square Roots

To be simplified, the radicand must have no perfect square factors other than 1.

Break the radicand into the product of a perfect square and another number, simplify by "removing" the square root of the perfect square.

Examples:

a. $\sqrt{27} = \sqrt{9 \cdot 3} = 3\sqrt{3}$ 9 is a perfect square that is a factor of 27. Its square root is 3 which is written on the outside of the radical sign.

b. $\sqrt{125} = \sqrt{25 \cdot 5} = 5\sqrt{5}$

c. $\sqrt{72} = \sqrt{36 \cdot 2} = 6\sqrt{2}$

Note: the last example could have also been simplified by the following series of steps:

$\sqrt{72} = \sqrt{9 \cdot 8} = 3\sqrt{8} = 3\sqrt{4 \cdot 2} = 6\sqrt{2}$

If you do not use the largest perfect square that is a factor at first, just keep simplifying until there are no more perfect square factors. If you "remove" another square root, multiply it by the number that is already outside the radical sign (when the $\sqrt{4}$ was "removed", it's value 2 was multiplied by the 3 that was already outside).

Note: Algebraic expressions with *even* exponents are perfect squares.

Examples:

a. $\sqrt{x^4} = x^2$ (because $x^2 \cdot x^2 = x^4$)

b. $\sqrt{x^8} = x^4$ (because $x^4 \cdot x^4 = x^8$)

Use this fact (in conjunction with the rules already learned) to simplify the following expressions:

a. $\sqrt{10x} \cdot \sqrt{8x} = \sqrt{80x^2} = \sqrt{16x^2 \cdot 5} = 4x\sqrt{5}$ First, the radicands are multiplied; next the radicand is broken into factors using one that is a perfect square, simplify.

b. $\sqrt{6x}\sqrt{3x^2} = \sqrt{18x^3} = \sqrt{9x^2 \cdot 2x} = 3x\sqrt{2x}$

Practice Problems: Simplify the following radical expressions:

1. $\sqrt{\mathbf{90}\boldsymbol{x}^{\mathbf{16}}\boldsymbol{y}^{\mathbf{8}}}$ **2.** $\sqrt{\mathbf{108}\boldsymbol{x}^{\mathbf{3}}\boldsymbol{y}^{\mathbf{2}}}$ **3.** $\sqrt{\mathbf{20}\boldsymbol{x}\boldsymbol{y}^{\mathbf{4}}}\sqrt{\mathbf{8}\boldsymbol{x}^{\mathbf{5}}}$

Solutions:

1. $\boldsymbol{x}^{\mathbf{8}}\boldsymbol{y}^{\mathbf{4}}\sqrt{\mathbf{10}}$ **2.** $\mathbf{6}\boldsymbol{xy}\sqrt{\mathbf{3}\boldsymbol{x}}$ **3.** $\mathbf{4}\boldsymbol{x}^{\mathbf{3}}\boldsymbol{y}^{\mathbf{2}}\sqrt{\mathbf{10}}$